

# Highly effective actions

---

**John H. Schwarz**

*California Institute of Technology,  
Pasadena, CA 91125, U.S.A.*

*E-mail:* [jhs@theory.caltech.edu](mailto:jhs@theory.caltech.edu)

**ABSTRACT:** It is conjectured that the world-volume action of a probe D3-brane in an  $AdS_5 \times S^5$  background of type IIB superstring theory, with one unit of flux, can be reinterpreted as the *exact* effective action (or highly effective action) for  $U(2)$   $\mathcal{N} = 4$  super Yang-Mills theory on the Coulomb branch. An analogous conjecture for  $U(2)_k \times U(2)_{-k}$  ABJM theory is also presented. The main evidence supporting these conjectures is that the brane actions have all of the expected symmetries and dualities. Highly effective actions have general coordinate invariance, even though they describe nongravitational theories.

**KEYWORDS:** Conformal Field Models in String Theory, D-branes, AdS-CFT Correspondence

**ARXIV EPRINT:** [1311.0305](https://arxiv.org/abs/1311.0305)

---

## Contents

|          |   |           |
|----------|---|-----------|
| <b>1</b> | <b>Introduction</b>   | <b>1</b>  |
| <b>2</b> | <b>The D3-brane in <math>AdS_5 \times S^5</math></b>        | <b>4</b>  |
| <b>3</b> | <b>ABJM theory</b>  | <b>10</b> |
| 3.1      | The M2-brane in $AdS_4 \times S^7$                          | 10        |
| 3.2      | The M2-brane in $AdS_4 \times S^7/Z_k$                      | 12        |
| 3.3      | The D2-brane in $AdS_4 \times CP^3$                         | 15        |
| 3.4      | Summary   | 17        |
| <b>4</b> | <b>The M5-brane in <math>AdS_7 \times S^4</math></b>        | <b>18</b> |
| <b>5</b> | <b>Conclusion</b>   | <b>20</b> |
| <b>A</b> | <b>The AdS Poincaré patch</b>                               | <b>22</b> |
| <b>B</b> | <b>S-duality of the D3-brane action</b>                     | <b>23</b> |
| <b>C</b> | <b>The D3-brane in global <math>AdS_5 \times S^5</math></b> | <b>27</b> |

---

## 1 Introduction

In the opening lecture at the Strings 2012 conference in Munich, I discussed some lessons learned in the course of my career. The one that is relevant here is: “Take coincidences seriously.” This principle has served me well on at least one previous occasion [1]. However, there have also been missed opportunities. For example, many of us were well aware of the fact that the isometry group of AdS space in  $d + 1$  dimensions is the same as the conformal group in  $d$  dimensions. This was generally assumed to be a strange, but surely irrelevant, coincidence. We now know better. In fact, the AdS/CFT correspondence will play a central role in the discussion that follows. However, the point of this paper is to emphasize a different coincidence and to suggest that it should be taken seriously.

This paper examines the construction of the world-volume theory of a single isolated  $p$ -brane in an  $AdS_{p+2} \times S^n$  background with  $N$  units of flux threading the sphere. The specific examples studied are the ones with maximal supersymmetry as well as ABJM theory, which has 3/4 maximal supersymmetry. The actions are of the usual type, consisting of a sum of two terms,  $S_1 + S_2$ , which is sometimes denoted  $S_{DBI} + S_{WZ}$ . The approximations that are implicit in probe-brane constructions are well-known. One of them is the probe approximation in which the effects of the probe brane on the background geometry and the gauge field configuration are neglected. This is tantamount to a large- $N$  approximation.

The other approximation is the assumption that world-volume fields, for example a Born-Infeld  $U(1)$  field strength, are slowly varying. This justifies excluding consideration of possible terms involving higher derivatives of fields. The field strength itself is allowed to be large. Despite these approximations, the formulas that are obtained in these constructions have a beautiful property: they fully incorporate the symmetry of the background as an exact global symmetry of the world-volume theory. This symmetry is the superconformal group  $PSU(2, 2|4)$  in the case of a D3-brane in  $AdS_5 \times S^5$ , for example. In this example it also includes the  $SL(2, \mathbb{Z})$  duality group, which is known to be an exact symmetry of type IIB superstring theory.

There was quite a bit of activity studying world-volume actions for branes in these geometries at the end of the last century. However, most of that work focused on superstrings, rather than higher-dimensional branes. An important example is the superstring in  $AdS_5 \times S^5$ , which was worked out in [2]. Also, those works that did study  $p$ -branes in  $AdS_{p+2}$  had different motivations from ours, and therefore made coordinate and gauge choices that are different from the ones made here. Our choices are specifically tuned to a particular goal: presenting the world-volume action of the brane in a form in which it can be interpreted as a candidate solution to an entirely different problem. That problem is the construction of the *effective action for a superconformal field theory on the Coulomb branch*. The fact that the world-volume theory of a  $p$ -brane in an AdS background is conformally invariant has been frequently noted, for example in [3]–[11]. The proposal that the result can be reinterpreted in the manner presented here motivates analyzing these brane actions in a very specific manner.

Let us discuss  $U(2) \mathcal{N} = 4$  super Yang-Mills theory to be specific. For most purposes it is correct to ignore a decoupled abelian multiplet and to speak of  $SU(N)$  rather than  $U(N)$ . However,  $U(N)$  is important for obtaining an  $SL(2, \mathbb{Z})$  duality group.<sup>1</sup> On the Coulomb branch the “photon” supermultiplet remains massless and the “ $W^\pm$ ” supermultiplets acquire a mass. In principle, the effective action on the Coulomb branch is obtained by performing the path integral over the massive fields, thereby “integrating them out” and producing a very complicated formula in terms of the massless photon supermultiplet only. If the computation could be done exactly, the resulting effective action would still encode the entire theory on the Coulomb branch. It would *not* be just a low-energy effective action. We propose to call such an effective action a *highly effective action* (HEA). Clearly, we do not have the tools to carry out such an exact computation, but in some cases (such as the one mentioned above) we do know many of the properties that the HEA should possess.

An important fact about the HEA is that it has all of the global symmetries of the original theory, though some of them are spontaneously broken. This is known to be true in the formulation with explicit  $W$  supermultiplets. It should continue to be true after they are integrated out, especially if one has the complete answer. The symmetry breaking is simply a consequence of assigning a vacuum expectation value (vev) to a massless scalar field.<sup>2</sup> As this vev goes to zero, the multiplets that have been integrated out become

<sup>1</sup>I am grateful to N. Seiberg for a discussion. See [12] for an explanation of the relevant issues.

<sup>2</sup>This field has sometimes been called the “dilaton.” We prefer not to use that terminology, because it is

massless and the formulas become singular. However, one does not need to specify the vev, so the action realizes the full symmetry. There are really only two discrete options, since the vev is the only scale in the problem. Either it is zero, or it can be set to one. In principle, the HEA contains all information about the theory, at energies both below and above the scale set by the vev. What one can question, indeed should question, is whether the procedure described in this paper gives the correct formula.

In the (known) formulation of the Coulomb branch theory, in which the massive  $W^\pm$  fields have not been integrated out, there are other massive particles (monopoles and dyons) that do not appear explicitly in the action. Yet the theory is fully specified at all scales. After the  $W^\pm$  have been integrated out, they should be on an equal footing with the monopoles and dyons, and one still has an exact characterization of the theory. In fact, the HEA should have soliton solutions, analogous to those considered in [13] (dubbed *BIONS* in [14]), that form complete  $SL(2, \mathbb{Z})$  multiplets. One of the advantages of the HEA description of the Coulomb branch is that dualities, such as  $SL(2, \mathbb{Z})$ , are much easier to understand than in the formulation with explicit  $W$  fields. We will argue that there are other advantages, as well.

Since  $\mathcal{N} = 4$  super Yang-Mills theory in the unbroken phase is conformal, we know that every term in the expansion of the Coulomb-branch effective action should have dimension four, which is the spacetime dimension of the field theory. There are no dimensionful parameters other than the vev, which does not need to be specified. The only surprising feature of the effective action (if you haven't thought about this before) is that inverse powers of a scalar field appear. Thus, individual terms can be arbitrarily complicated and yet have their dimensions end up as four simply by including appropriate (inverse) powers of the scalar field.

The action that we will obtain for a suitably placed  $p$ -brane in a background geometry containing an  $AdS_{p+2}$  factor has all the symmetries and other properties that a Coulomb branch effective action should have. Therefore, this action provides a compelling candidate for the HEA. However, there are some caveats. First, it is not obvious to what extent symmetries and other properties determine the solution. So this could be a wrong formula that just happens to have many correct properties, thereby demonstrating that taking coincidences too seriously can lead you astray. The general expectation is that the larger the superconformal symmetry group that the action should incorporate, the more limited the possibilities are. When there are additional requirements, such as dualities,<sup>3</sup> they can greatly strengthen the case. A second caveat is that the solutions that are obtained by brane constructions depend on an integer parameter,  $N$ , which is the number of units of background flux through the sphere. We will find that the brane action satisfies all of the symmetry and duality requirements for any choice of  $N$ , so there is at least this much

---

unrelated to the string theory dilaton, which controls the coupling constant. In fact, in a brane construction this scalar field corresponds to the radial coordinate of the AdS space.

<sup>3</sup>We do not call the  $SL(2, \mathbb{Z})$  dualities ‘symmetries,’ because they relate the theory at different values of the coupling constants. In some settings, they even relate different theories.  $SL(2, \mathbb{Z})$  is a symmetry of type IIB superstring theory, which is spontaneously broken by the choice of vacuum. For specific values of the modulus a  $\mathbb{Z}_2$  or  $\mathbb{Z}_3$  subgroup can survive as a symmetry.

nonuniqueness. Even though the probe approximation is only valid for large  $N$ , the choice  $N = 1$  is the most natural candidate for the HEA of the  $U(2)$  gauge theory. We will ignore the decoupled  $U(1)$  multiplet, which was discussed earlier. It needs to be adjoined to the brane-probe action.

This paper only discusses the bosonic degrees of freedom. This allows us to emphasize conceptual issues with a minimum of distracting technicalities. Also, it is important to have these formulas completely debugged, before confronting the more complicated formulas with fermions. The ultimate inclusion of the fermi fields is essential for the results to be truly meaningful, so that is an important project for the future. However, the formulas presented here already are of interest, since they are the bosonic truncations of the HEAs that include fermi fields. Some of the formulas presented here, such as the bosonic part of the D3-brane action, have appeared previously in the literature. However, we have clarified a few details, and set the stage for inclusion of the fermions in a form appropriate for reinterpretation as a Coulomb-branch effective action.

The local symmetries of the world-volume theory, which are general coordinate invariance and kappa symmetry (when fermions are included), provide crucial constraints in the construction of  $p$ -brane world-volume actions. There is a natural gauge choice that brings the gauge-invariant formulas into a recognizable form containing only the expected supermultiplet of fields. The local symmetries are no longer apparent in the gauge-fixed theory, but they control some of the symmetry transformations of the resulting theory. The supersymmetry transformations, for example, take a rather simple form before gauge fixing. They become much more complicated after gauge fixing, because they then include compensating local kappa transformations required to maintain the gauge choice. More mundane examples of this are already present in the bosonic truncations described in this paper. Thus, an understanding of the local symmetries gives a systematic procedure to derive how symmetry transformations of the HEA incorporate quantum effects attributable to the fields that have been integrated out. The fact that these nongravitational theories are naturally formulated with general coordinate invariance is quite intriguing.

Section 2 describes the construction of the HEA that corresponds to the world-volume theory of a probe D3-brane in  $AdS_5 \times S^5$ . Section 3 describes analogous M2-brane and D2-brane constructions of the HEA for  $U(2)_k \times U(2)_{-k}$  ABJM theory on the Coulomb branch. The two descriptions are shown to be related by a duality transformation, analogous to the S-duality of the four-dimensional theory of section 2, which proves their equivalence. Section 4 describes the M5-brane in  $AdS_7 \times S^4$ . The analysis in each case utilizes the Poincaré-patch description of anti de Sitter spacetime. The Poincaré patch is reviewed in appendix A. Appendix B demonstrates the S-duality of the D3-brane HEA. Appendix C discusses an alternative to the Poincaré patch, namely the geodesically complete covering space, and explains why it is not appropriate for our purposes.

## 2 The D3-brane in $AdS_5 \times S^5$

Type IIB superstring theory has a maximally supersymmetric solution with  $PSU(2, 2|4)$  isometry. In this paper we only consider bosonic degrees of freedom, so the relevant part of the

symmetry is the bosonic subgroup  $SO(4, 2) \times SO(6)$ . The solution has the ten-dimensional geometry  $AdS_5 \times S^5$ , where both factors have radius  $R$ . Also,  $N$  units of five-form flux  $F_5 = dC_4$  thread the five-sphere. Since  $F_5$  is self dual, it follows that it is proportional to  $\text{vol}(AdS_5) + \text{vol}(S^5)$ , with a constant of proportionality to be discussed later.

Using the Poincaré patch coordinates discussed in appendix A, the ten-dimensional metric is

$$ds^2 = R^2 (v^2 dx \cdot dx + v^{-2} dv^2 + d\Omega_5^2) = R^2 (v^2 dx \cdot dx + v^{-2} dv \cdot dv). \quad (2.1)$$

$v$  is now the length of the six-vector  $v^I$ , i.e.,  $v^2 = \sum (v^I)^2 = v \cdot v$ . This is an  $SO(6)$  invariant inner product. We can also check the conformal symmetry of the  $AdS_5$  volume form

$$\text{vol}(AdS_5) = R^5 e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge f. \quad (2.2)$$

Using eqs. (A.7) and (A.8), it is clear that  $\delta f$  gives a vanishing contribution to  $\delta \text{vol}(AdS_5)$  and that the contribution of  $\delta e^\mu$  is proportional to  $\text{tr}(b^\mu x_\nu - x^\mu b_\nu) = 0$ .

The coordinates  $v^I$  will be identified as the world-volume scalar fields of the D3-brane. However, as introduced here, they would not be nicely normalized. Therefore, let us define  $v^I = \sqrt{c_3} \phi^I$ , where  $c_3$  is a dimensionless constant that will be chosen later to give the desired normalization. The metric becomes

$$ds^2 = g_{MN} dx^M dx^N = R^2 (c_3 \phi^2 dx \cdot dx + \phi^{-2} d\phi \cdot d\phi). \quad (2.3)$$

where  $x^M = (x^\mu, \phi^I)$ ,  $g_{\mu\nu} = c_3 R^2 \phi^2 \eta_{\mu\nu}$ ,  $g_{IJ} = R^2 \phi^{-2} \delta_{IJ}$ . In these coordinates  $\phi = \infty$  is the boundary of AdS and  $\phi = 0$  is the Poincaré-patch horizon.

The radius  $R$ , raised to the fourth power, is determined by the ten-dimensional type IIB superstring theory solution to be

$$R^4 = 4\pi g_s N (\alpha')^2. \quad (2.4)$$

The Regge slope  $\alpha'$  is given by  $\alpha' = l_s^2$ , where  $l_s$  is the string length scale. Here we are interested in ten dimensions. However, in  $D$  dimensions Newton's constant,  $G_D$ , is given by  $32\pi^2 G_D = (2\pi l_p)^{D-2} = g_s^2 (2\pi l_s)^{D-2}$ , where  $l_p$  is the  $D$ -dimensional Planck length.

The D3-brane world-volume action is given as the sum of two terms  $S = S_1 + S_2$ .  $S_1$  has a Dirac/Born-Infeld/Nambu-Goto type structure. When fermions are included (for the case of a flat-spacetime background), it also has a Volkov-Akulov structure.  $S_2$  has a Chern-Simons/Wess-Zumino type structure.<sup>4</sup>

The standard formula for the bosonic part of  $S_1$  for a D-brane is a functional of the embedding functions  $x^M(\sigma^\alpha)$  and a world-volume  $U(1)$  gauge field  $A_\beta(\sigma^\alpha)$  with field strength  $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ :

$$S_1 = -T_{D3} \int \sqrt{-\det(G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})} d^4\sigma, \quad (2.5)$$

---

<sup>4</sup>Moreover, when fermions are included,  $S_1$  and  $S_2$  are related by local kappa symmetry, which was discovered for the massless superparticle of [15], which has no  $S_2$  term, in [16] and utilized for the superstring, which does have an  $S_2$  term, in [17].

where the D3-brane tension is

$$T_{D3} = \frac{2\pi}{g_s(2\pi l_s)^4}. \quad (2.6)$$

Combining eqs. (2.4) and (2.6),

$$R^4 T_{D3} = \frac{N}{2\pi^2}. \quad (2.7)$$

Also,

$$2\pi\alpha'/R^2 = \sqrt{\pi/g_s N}. \quad (2.8)$$

In general, the  $S_1$  integrand would contain a factor  $e^{-\Phi}$ , where  $\Phi$  is the ten-dimensional dilaton field. However, in the background under consideration this factor is a constant,  $1/g_s$ , which is included in  $T_{D3}$ .  $G_{\alpha\beta}$  is the induced four-dimensional world-volume metric

$$G_{\alpha\beta} = g_{MN}(x)\partial_\alpha x^M \partial_\beta x^N. \quad (2.9)$$

The action in eq. (2.5) has four-dimensional general coordinate invariance, since the integrand transforms as a scalar density. Furthermore, the conformal symmetry group is realized as a global symmetry, since  $G_{\alpha\beta}$  and  $F_{\alpha\beta}$  are separately invariant under the entire conformal group. The previous analysis ensures this for  $G_{\alpha\beta}$ . In the case of  $F_{\alpha\beta}$  it is a triviality: the gauge field  $A_\alpha$  is inert under the entire conformal group. It will transform nontrivially after a gauge choice is made.

Substituting the ten-dimensional metric  $g_{MN}$  given previously, pulling out some factors from the square root, and using eqs. (2.4) and (2.7), we obtain

$$S_1 = -\frac{N}{2\pi^2} \int \phi^4 \sqrt{-\det \left( c_3 \partial_\alpha x \cdot \partial_\beta x + \phi^{-4} \partial_\alpha \phi \cdot \partial_\beta \phi + \sqrt{\pi/(g_s N)} \phi^{-2} F_{\alpha\beta} \right)} d^4 \sigma. \quad (2.10)$$

The next step is to use the general coordinate invariance symmetry to choose a convenient gauge. The natural and appropriate choice for our purposes is static gauge. This gauge identifies the world-volume coordinates  $\sigma^\alpha$  with the spacetime coordinates  $x^\mu$ :

$$x^\mu(\sigma) = \delta^\mu_\alpha \sigma^\alpha. \quad (2.11)$$

Then the action takes the form

$$S_1 = -\frac{N c_3^2}{2\pi^2} \int \phi^4 \sqrt{-\det \left( \eta_{\mu\nu} + \frac{\partial_\mu \phi \cdot \partial_\nu \phi}{c_3 \phi^4} + \sqrt{\pi/(g_s N)} \frac{F_{\mu\nu}}{c_3 \phi^2} \right)} d^4 x. \quad (2.12)$$

Having chosen the static gauge, the fields  $\phi^I$  and  $A_\mu$  become functions of  $x^\mu$ . The four-dimensional spacetime metric is simply the flat Minkowski metric denoted  $\eta_{\mu\nu}$ .

If one expands out the square root in eq. (2.12) in powers of  $\partial\phi$  and  $F$ , the leading term is proportional to  $\int \phi^4 d^4 x$ . In his famous paper [3], Maldacena explained that this term must be canceled, since such a term would imply that a force acts on the brane in the radial direction. We know that we are dealing with a system of parallel BPS branes at rest, for which there must be a perfect cancellation of forces. Thus, even though this term is conformally invariant and  $SO(6)$  invariant, it should not appear. We will demonstrate



below that it cancels against a contribution from  $S_2$ . So the requisite cancellation will arise naturally without any need to introduce it in an *ad hoc* manner.

The next terms in the expansion of the Lagrangian are the kinetic terms of the free theory

$$L_{\text{free}} = -\frac{Nc_3}{4\pi^2}\partial_\mu\phi\cdot\partial^\mu\phi - \frac{1}{8\pi g_s}F_{\mu\nu}F^{\mu\nu}. \quad (2.13)$$

Requiring that  $\phi^I$  is canonically normalized leads to the choice  $c_3 = 2\pi^2/N$ . A better alternative is

$$c_3 = \frac{\pi}{g_s N}. \quad (2.14)$$

For this choice the entire action depends only on the 't Hooft parameter  $\lambda = g_s N$ , aside from an overall factor of  $N$  (or  $1/g_s$ ). This structure suggests the conjecture that the loop expansion of this action (after we include  $S_2$  and fermions) corresponds to the string theory loop expansion, or equivalently to the genus (or  $1/N$ ) expansion at fixed  $\lambda$ . This conjecture is somewhat uncertain, because we have not proposed a precise interpretation of the formula with  $N > 1$ . We will comment on this issue in the conclusion.

The conjecture that the loop expansion of the  $N = 1$  HEA corresponds to the topological expansion of the super Yang-Mills theory could be wrong even if our main conjecture, namely that we have found the HEA for the  $U(2)$  theory, is correct. However, if this secondary conjecture is also correct, there would be interesting consequences. For one thing, it would imply that the HEA action (after adding the contributions of fermions and of  $S_2$ ) has dual superconformal symmetry when treated classically, and hence the full Yangian symmetry. It should be very interesting to compute scattering amplitudes in the tree approximation to determine whether they are compatible with this symmetry. This is a very clean problem, since there are no integrals to evaluate and no infrared divergences to regulate. There could be beautiful formulas waiting to be discovered. One could also explore Wilson loops.

An important fact is that the construction described here ensures that the action is conformally invariant and  $SO(6)$  invariant. (When fermions are included the full  $PSU(2, 2|4)$  superconformal symmetry will be built in.) The conformal symmetry and the  $SO(6)$  symmetry are both spontaneously broken when one assigns a nonzero expectation value  $\langle\phi^I\rangle$ . This ensures that the inverse powers of  $\phi$  are well-defined. Physically,  $\langle\phi^I\rangle$  describes the position of the brane in the radial direction of  $AdA_5$  and on the  $S^5$ . In the limit that  $\langle\phi\rangle$  goes to zero, the brane approaches the horizon, and new massless degrees of freedom arise. This is the significance of the singularities at  $\phi = 0$ .

Having fixed the static gauge, the formulas for infinitesimal conformal transformations are modified by the addition of a compensating general coordinate transformation, which is required to maintain the gauge choice. Denoting the new transformation by  $\Delta$  and the old one by  $\delta$ , we have

$$\Delta x^\mu = \delta x^\mu + \xi^\mu = 0. \quad (2.15)$$

$$\Delta\phi^I = \delta\phi^I + \xi^\mu\partial_\mu\phi^I \quad (2.16)$$

$$\Delta A_\mu = \xi^\nu F_{\nu\mu}. \quad (2.17)$$



Equations (A.4) and (2.15) give

$$\xi^\mu = -b^\mu \left( \frac{1}{c_3 \phi^2} + x \cdot x \right) + 2b \cdot x x^\mu. \quad (2.18)$$

Hence, with this value of  $\xi^\mu$ , the compensated conformal transformations of the remaining bosonic fields in the static gauge are

$$\Delta \phi^I = 2b \cdot x \phi^I + \xi^\mu \partial_\mu \phi^I \quad (2.19)$$

$$\Delta A_\mu = \xi^\nu F_{\nu\mu}. \quad (2.20)$$

When fermions are included, there will be an analogous analysis of supersymmetry transformations involving compensating local kappa transformations. From the point of view of the effective action, these compensating gauge transformation contributions to global symmetry transformations correspond to quantum corrections that arise from integrating out the massive  $W$  supermultiplets.

The second term in the D3-brane action,  $S_2$ , is an integral of a 4-form whose bosonic part has terms of the form  $C_4$ ,  $C_0 F \wedge F$ ,  $C_0 \text{tr}(R \wedge R)$  with coefficients to be discussed later. The two-forms  $B_2$  and  $C_2$  do not appear, because they vanish in the  $AdS_5 \times S^5$  background. Here  $F$  is the abelian world-volume field strength, which already appeared in  $S_1$ .  $C_4$  is the RR 4-form whose field strength,  $F_5 = dC_4$ , is self dual.  $R$  is the ten-dimensional spacetime curvature two-form, pulled back to the D3-brane world volume. Fortunately,  $\text{tr}(R \wedge R)$  vanishes in the  $AdS_5 \times S^5$  background, so we will not discuss it further.  $C_0$  is the RR 0-form, which can take an arbitrary constant value  $\langle C_0 \rangle = \chi = \theta/2\pi$  in the  $AdS_5 \times S^5$  background. The parameter  $\theta$  will correspond to the usual theta angle of the gauge theory. The theory is expected to be invariant under  $\chi \rightarrow \chi + 1$ , which is the T transformation of the  $SL(2, \mathbb{Z})$  duality group.

The self-dual five-form field  $F_5$  in the  $AdS_5 \times S^5$  background is proportional to the sum of the volume form for the  $S^5$  and its dual, which is the volume form for the  $AdS_5$ . Thus, the field strength is  $F_5 = k_3(\text{vol}(S^5) + \text{vol}(AdS_5))$ , where the coefficient  $k_3$  depends on normalization conventions. Referring to eq. (A.3), we read off the volume form for the unit radius  $AdS_5$  space as the wedge product of five one-forms

$$\text{vol}(AdS_5) = v^3 dv \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = d \left( \frac{v^4}{4} dx^0 \wedge \dots \wedge dx^3 \right). \quad (2.21)$$

We can now write the  $C_4$  part of  $S_2$  in the form

$$\mu_3 \int_{\partial M} C_4 = \mu_3 k_3 \int_M (\text{vol}(S^5) + \text{vol}(AdS_5)) = \mu_3 k_3 \int_M \text{vol}(AdS_5). \quad (2.22)$$

Here  $M$  denotes a five-dimensional region whose boundary is the D3-brane world volume. There is a natural choice for this region. Recall that the fields  $v^I$  determine the radial coordinate  $v$  and the position on the five-sphere of the D3-brane. If we choose the region  $M$  to be the region between 0 and  $v$ , we can pass from the five-form to the four-form by integrating  $v'$  from 0 to  $v$ . The upper limit gives the desired result and the lower limit contributes zero. Thus,

$$\mu_3 k_3 \int_M \text{vol}(AdS_5) = \frac{\mu_3 k_3}{4} \int_{\partial M} v^4 dx^0 \wedge \dots \wedge dx^3 = \frac{\mu_3 k_3 c_3^2}{4} \int \phi^4 d^4 x. \quad (2.23)$$

While this is a correct formula, one should understand that

$$d^4x = \sqrt{-\det(\eta_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu)}d^4\sigma. \quad (2.24)$$

In static gauge, the Jacobian factor is equal to one, and  $d^4\sigma = d^4x$ .

The next step is to evaluate  $\mu_3k_3$ . The way to do this is to use the fact that there are  $N$  units of five-form flux. This means that the *same* five-form,  $\mu_3F_5$ , integrated over an  $S^5$  of unit radius is  $2\pi N$ :

$$\mu_3k_3 \int_{S^5} (\text{vol}(S^5) + \text{vol}(AdS_5)) = \mu_3k_3 \int_{S^5} \text{vol}(S^5) = \mu_3k_3\pi^3 = 2\pi N. \quad (2.25)$$

Substituting  $\mu_3k_3c_3 = 4$  in eq. (2.23), and restoring the  $\chi$  term, we obtain

$$S_2 = \frac{Nc_3^2}{2\pi^2} \int \phi^4 d^4x + \frac{\chi}{8\pi} \int F \wedge F. \quad (2.26)$$

Combining this with  $S_1$  in eq. (2.12) gives the total result.

$$S = -\frac{Nc_3^2}{2\pi^2} \int \phi^4 \left( \sqrt{-\det \left( \eta_{\mu\nu} + \frac{\partial_\mu \phi \cdot \partial_\nu \phi}{c_3 \phi^4} + \sqrt{\pi\tau_2/N} \frac{F_{\mu\nu}}{c_3 \phi^2} \right)} - 1 \right) d^4x + \frac{\tau_1}{8\pi} \int F \wedge F. \quad (2.27)$$

Here we have introduced the parameter

$$\tau = \tau_1 + i\tau_2 = \langle C_0 + i \exp(-\Phi) \rangle = \chi + i/g_s. \quad (2.28)$$

For the specific choice  $c_3 = \pi/\lambda$ , where  $\lambda = g_s N$ ,

$$S = -\frac{1}{2g_s\lambda} \int \phi^4 \left( \sqrt{-\det \left( \eta_{\mu\nu} + \frac{\lambda}{\pi} \frac{\partial_\mu \phi \cdot \partial_\nu \phi}{\phi^4} + \sqrt{\frac{\lambda}{\pi}} \frac{F_{\mu\nu}}{\phi^2} \right)} - 1 \right) d^4x + \frac{\chi}{8\pi} \int F \wedge F. \quad (2.29)$$

As promised earlier, there is a precise cancellation of the undesired  $\int \phi^4 d^4x$  terms, which arises from combining  $S_1$  and  $S_2$ . Had we considered an anti-D3-brane (a  $\overline{\text{D3}}$ -brane) instead,  $S_2$  would have had the opposite sign, and the cancellation of forces would not have occurred. Rather, there would be a potential  $V \sim \phi^4$ , which implies that the  $\overline{\text{D3}}$ -brane would be attracted to  $\phi = 0$ , which is the Poincaré-patch horizon. In the Coulomb branch interpretation, this is the origin of the moduli space, the singular point at which fields that have been integrated out become massless, and the nonabelian gauge symmetry is restored. It is known that a potential term in a Coulomb-branch effective action is not consistent with  $\mathcal{N} = 4$  supersymmetry [18]. It is also known that the world-volume theory of a  $\overline{\text{D3}}$ -brane probe is not supersymmetric.

In appendix B we demonstrate that the S-duality group of eq. (2.29) is precisely  $\text{SL}(2, \mathbb{Z})$  for all  $N$ . Under the S transformation of the duality group, the gauge field  $A_\mu$  is replaced by a dual gauge field and  $\tau \rightarrow -1/\tau$ . No proof is known for the formulation of the theory with explicit  $W$  supermultiplets. It is truly remarkable that this complicated nonlinear formula, whose essential structure was introduced by Born and Infeld almost 80

years ago [19], should have this symmetry. This fact provides further confirmation that the action derived by studying a probe D3-brane has all the properties required for the effective action for the  $U(2)$   $\mathcal{N} = 4$  super Yang-Mills theory on the Coulomb branch. The only freedom is the choice of the integer parameter  $N$ . The choice  $N = 1$  is probably the correct one. The actions with  $N > 1$  may play a role in the construction of effective field theories for higher-rank gauge groups, but there must be additional ingredients as well.

As mentioned in the introduction, this HEA should have soliton solutions that form a complete  $SL(2, \mathbb{Z})$  multiplet. These will describe the  $W$  supermultiplets that have been integrated out as well as monopole and dyon supermultiplets. From a string theory viewpoint, these can be interpreted as  $(p, q)$  strings ending on the D3-brane. There should also be instanton solutions of the Euclideanized theory. These can be interpreted as embedded  $D(-1)$ -branes.

Appendix C analyzes an analogous problem in which a D3-brane of  $S^3$  topology probes global  $AdS_5 \times S^5$ . In this case it is shown that the contributions to the potential from  $S_1$  and  $S_2$  do not cancel. As a result, the brane probe is unstable to decay and shrinking to a point at the center of global  $AdS_5$ . In terms of the nonabelian gauge theory on  $S^3$ , the interpretation is that there is no Coulomb branch. We will not repeat the analysis of this appendix for the brane theories that we consider in the next sections. Only the Poincaré-patch descriptions will be presented.

### 3 ABJM theory

The purpose of this section is to derive a candidate formula for the HEA for  $U(2)_k \times U(2)_{-k}$  ABJM theory [27] on the Coulomb branch. A formula will be derived first by considering an M2-brane in 11 dimensions. Then an alternative formula will be obtained by considering a type IIA D2-brane in  $AdS_4 \times CP^3$ . The equivalence of the two formulas will be demonstrated. The D2-brane version of the formula has the field content and structure of an abelian  $\mathcal{N} = 6$  super Yang-Mills theory, whereas the M2-brane version has the field content and structure of a  $U(1)_k \times U(1)_{-k}$  ABJM theory i.e., an  $\mathcal{N} = 6$  superconformal Chern-Simons theory. All calculations will be carried out for  $N$  units of flux. However, as before, we conjecture that  $N = 1$  is the relevant choice.

#### 3.1 The M2-brane in $AdS_4 \times S^7$

This geometry corresponds to the  $k = 1$  special case of the more general geometry  $AdS_4 \times S^7/\mathbb{Z}_k$ , which is considered in the next subsection. In the M2-brane case the radius of the  $AdS_4$  is half the radius  $R$  of the  $S^7$ , and the coordinate  $x^\mu$  is a 3-vector. So, in terms of Poincaré-patch coordinates for the AdS factor, the 11-dimensional metric is

$$ds^2 = (R/2)^2 (v^2 dx \cdot dx + v^{-2} dv^2) + R^2 d\Omega_7^2 = R^2 (c_2 \phi^4 dx \cdot dx + \phi^{-2} d\phi^2 + d\Omega_7^2). \quad (3.1)$$

The change of variables  $v = 2\sqrt{c_2}\phi^2$  is motivated by the fact that  $v$  has dimension 1, whereas a scalar field in three dimensions should have dimension  $1/2$ . As in the D3-brane case, the constant  $c_2$  is included so that the normalization of the scalar fields can be chosen

later. The metric can be rewritten in the form

$$ds^2 = R^2 (c_2 \phi^4 dx \cdot dx + \phi^{-2} d\phi^I d\phi^I), \quad (3.2)$$

where  $\phi^I$  is an eight-component vector of length  $\phi$ . The  $AdS_4 \times S^7$  solution of 11-dimensional supergravity, with  $N$  units of flux through the seven-sphere, relates the radius  $R$  to the 11-dimensional Planck length  $l_p$  by

$$R^6 = 32\pi^2 N l_p^6. \quad (3.3)$$

There is no world-volume gauge field in this case, so the  $S_1$  term in the action is just

$$S_1 = -T_{M2} \int \sqrt{-\det(G_{\alpha\beta})} d^3\sigma, \quad (3.4)$$

where  $G_{\alpha\beta}$  is the pullback of the 11-dimensional spacetime metric. Since  $T_{M2} = (4\pi^2 l_p^3)^{-1}$ ,

$$T_{M2} R^3 = \sqrt{2N}/\pi. \quad (3.5)$$

Choosing static gauge  $x^\mu(\sigma) = \delta^\mu_\alpha \sigma^\alpha$ , as before, yields

$$S_1 = -T_{M2} \int \sqrt{-\det(c_2 R^2 \phi^4 \eta_{\mu\nu} + (R/\phi)^2 \partial_\mu \phi^I \partial_\nu \phi^I)} d^3x, \quad (3.6)$$

which can be rewritten as

$$S_1 = -\frac{\sqrt{2N}}{\pi} c_2^{3/2} \int \phi^6 \sqrt{-\det\left(\eta_{\mu\nu} + \frac{\partial_\mu \phi^I \partial_\nu \phi^I}{c_2 \phi^6}\right)} d^3x. \quad (3.7)$$

Now we turn to  $S_2 = \mu_2 \int A_3$ . Here  $A_3$  is the pullback of the M-theory three-form, whose field strength is denoted  $F_4 = dA_3$ . This field strength is proportional (with a coefficient denoted  $k_2$ ) to the volume form for  $AdS_4$ . Therefore

$$F_4 = k_2 c_2^{3/2} R^4 \phi^5 d\phi \wedge dx^0 \wedge dx^1 \wedge dx^2 = \frac{1}{6} k_2 c_2^{3/2} R^4 d(\phi^6 dx^0 \wedge dx^1 \wedge dx^2). \quad (3.8)$$

Thus, by the same reasoning as in the D3-brane analysis, we obtain

$$S_2 = \frac{\mu_2 k_2 c_2^{3/2} R^4}{6} \int \phi^6 d^3x. \quad (3.9)$$

We can now determine  $\mu_2 k_2$  by requiring that  $\mu_2 \int_{S^7} \star F_4 = 2\pi N$ . This is the statement that there are  $N$  units of flux threading the  $S^7$ . To avoid ambiguity (and errors) in relating the normalization of  $F_4$  to its Hodge dual, it is important that the Hodge dual operation be applied to a dimensionless expression. Since  $F_4$  is not dimensionless to begin with, this requires the choice of a basic unit of length. We find that to get the desired result, the appropriate choice for this unit of length is  $2\pi l_p$ . It would be desirable to have a better explanation for this rule, but this is the best we can do without discussing fermions (aside from the physical argument that there should be no force on the brane). When fermions are included, kappa symmetry will relate  $S_2$  to  $S_1$  and determine this

normalization unambiguously. This issue did not arise in the D3-brane problem, because  $F_5$  was self-dual.

Supplying the appropriate power of  $2\pi l_p$ , we have

$$\star F_4 = \frac{k_2 R^7}{(2\pi l_p)^3} \text{vol}(S^7), \quad (3.10)$$

where  $\text{vol}(S^7)$  denotes the volume form for a unit radius seven-sphere. Such a sphere has volume  $\pi^4/3$ , and thus

$$\mu_2 \int_{S^7} \star F_4 = \frac{\mu_2 k_2 R^7}{(2\pi l_p)^3} \frac{\pi^4}{3} = 2\pi N. \quad (3.11)$$

It follows that the coefficient in  $S_2$  is

$$\frac{\mu_2 k_2 R^4}{6} = \frac{2\pi N (2\pi l_p)^3}{2\pi^4 R^3} = \frac{\sqrt{2N}}{\pi}. \quad (3.12)$$

This is exactly what we want so that

$$S = S_1 + S_2 = -\frac{\sqrt{2N}}{\pi} c_2^{3/2} \int \phi^6 \left[ \sqrt{-\det \left( \eta_{\mu\nu} + \frac{\partial_\mu \phi^I \partial_\nu \phi^I}{c_2 \phi^6} \right)} - 1 \right] d^3x. \quad (3.13)$$

As in the D3-brane problem, we find cancellation of the nonderivative terms, which would otherwise give a net force on the brane. As before, the sign of  $S_2$  would be reversed for an anti-brane. Then there would be a potential  $V \sim \phi^6$ , which implies that the anti-brane is attracted to  $\phi = 0$ , the origin of the moduli space (and the horizon of the AdS space).

Finally, we can determine the parameter  $c_2$  by requiring that the  $\phi$  kinetic term is normalized in some particular way. For example, requiring that it is canonically normalized gives  $c_2 = \pi^2/(2N)$ . We will make a different choice later.

### 3.2 The M2-brane in $\text{AdS}_4 \times S^7/\mathbb{Z}_k$

The next case we wish to consider should correspond to the ABJM conformal field theory. In particular, we wish to derive a candidate HEA for  $U(2)_k \times U(2)_{-k}$  superconformal gauge theory on the Coulomb branch. This theory has  $\mathcal{N} = 6$  (3/4 maximal) supersymmetry (for  $k > 2$ ). There are two integers in the problem,  $N$  is the number of units of flux, as before. The second integer,  $k$ , controls the levels of the Chern-Simons terms in the superconformal Chern-Simons theory. The dual 11-dimensional geometry is closely related to that discussed for  $\text{AdS}_4 \times S^7$ , which corresponds to  $k = 1$ . The only difference is that the seven-sphere is modded out by the discrete group  $\mathbb{Z}_k$ . So we need to figure out how to modify the previous analysis to describe this situation. One thing is obvious: we should replace the eight real scalar fields  $\phi^I$  by four complex scalar fields  $\Phi^A$ :

$$\Phi^A = \phi^A + i\phi^{A+4} \quad A = 1, 2, 3, 4, \quad (3.14)$$

so that

$$\Phi^2 = \sum_{A=1}^4 |\Phi^A|^2 = \sum_{I=1}^8 (\phi^I)^2 = \phi^2. \quad (3.15)$$

Only a  $U(4)$  subgroup of  $SO(8)$  will survive. Moreover, only the  $SU(4)$  part of this is an R-symmetry group belonging to the superconformal symmetry group  $OSp(6|4)$ .

One may be tempted to incorporate two  $U(1)$  gauge fields into the description. However, as discussed in section 2.3 of the ABJM paper, there is a simpler option. One can add a single periodic real scalar coordinate  $\tau$ , with period  $2\pi$ , and a compensating local symmetry. The metric in eq. (3.2) is modified to become

$$ds^2 = R^2 (c_2 \Phi^4 dx \cdot dx + \Phi^{-2} D\Phi \cdot D\bar{\Phi}), \quad (3.16)$$

where

$$D\Phi^A = d\Phi^A + ik^{-1}\Phi^A d\tau \quad \text{and} \quad D\bar{\Phi}_A = d\bar{\Phi}_A - ik^{-1}\bar{\Phi}_A d\tau. \quad (3.17)$$

Equivalently, defining a one-form  $B = k^{-1}d\tau$ ,

$$D\Phi^A = (d + iB)\Phi^A \quad \text{and} \quad D\bar{\Phi}_A = (d - iB)\bar{\Phi}_A. \quad (3.18)$$

The radius is now given by

$$R^6 = 32\pi^2 k N l_p^6. \quad (3.19)$$

This metric is invariant under the local symmetry  $\Phi^A \rightarrow \exp(i\theta)\Phi^A$  and  $\tau \rightarrow \tau - k\theta$ , since these imply that  $D\Phi^A \rightarrow \exp(i\theta)D\Phi^A$ . One could use this “gauge freedom” to set  $\tau = 0$ , but this would still leave transformations with  $\theta$  an integer multiple of  $2\pi/k$ . In other words, one is left with the equivalence

$$\Phi^A \sim e^{2\pi i/k} \Phi^A. \quad (3.20)$$

Thus, the resulting geometry is the desired  $AdS_4 \times S^7/Z_k$ . The following discussion uses the gauge invariant formulation. Equivalently, we can set  $\tau = k\sigma$  and  $B_\mu = \partial_\mu\sigma$ , provided we enforce a  $2\pi/k$  periodicity in  $\sigma$  by means of a suitable Chern-Simons term. The only other significant modification of previous section concerns the flux quantization condition, which now becomes  $\mu_2 \int_{S^7/Z_k} \star F_4 = 2\pi N$ . This is accounted for by replacing  $N$  by  $kN$  in eqs. (3.3) and (3.11). This results in

$$S = S_1 + S_2 = -\frac{\sqrt{2kN}}{\pi} c_2^{3/2} \int \Phi^6 \left[ \sqrt{-\det \left( \eta_{\mu\nu} + \frac{\text{Re} [D_\mu \Phi^A D_\nu \bar{\Phi}_A]}{c_2 \Phi^6} \right)} - 1 \right] d^3x. \quad (3.21)$$

The kinetic term for the scalar fields is then

$$S_{\text{kin}} = -\frac{k}{\pi} \sqrt{\frac{c_2 \lambda}{2}} \int \text{Re} [D_\mu \Phi^A D_\nu \bar{\Phi}_A] d^3x, \quad (3.22)$$

where we have introduced the 't Hooft parameter

$$\lambda = N/k. \quad (3.23)$$

The quantity  $D\Phi^A D\bar{\Phi}_A$  can be recast as follows

$$D\Phi^A D\bar{\Phi}_A = d\Phi^A d\bar{\Phi}_A + 2\Phi^2 BW + \Phi^2 B^2 = d\Phi^A d\bar{\Phi}_A - \Phi^2 W^2 + \Phi^2 (B + W)^2, \quad (3.24)$$

where

$$W = \Phi^{-2} \text{Im} [\Phi^A d\bar{\Phi}_A]. \quad (3.25)$$

These formulas enable us to recast the determinant in the action in the form

$$\Delta = \det \left( G_{\mu\nu} + \frac{(B_\mu + W_\mu)(B_\nu + W_\nu)}{c_2 \Phi^4} \right), \quad (3.26)$$

where

$$G_{\mu\nu} = \eta_{\mu\nu} + \frac{\text{Re} [\partial_\mu \Phi^A \partial_\nu \bar{\Phi}_A]}{c_2 \Phi^6} - \frac{\text{Im} [\Phi^A \partial_\mu \bar{\Phi}_A] \text{Im} [\Phi^A \partial_\nu \bar{\Phi}_A]}{c_2 \Phi^8}. \quad (3.27)$$

Because the second term inside the determinant is now rank one, we can recast the square root of the determinant in the form

$$\sqrt{-\Delta} = \sqrt{-G} \sqrt{1 + \frac{(B+W)^2}{c_2 \Phi^4}}, \quad (3.28)$$

where  $(B+W)^2 = G^{\mu\nu}(B_\mu + W_\mu)(B_\nu + W_\nu)$ . (Inner products in this paper use the Lorentz metric unless otherwise specified.) This has successfully isolated the  $B$  dependence, which was the purpose of these maneuvers. The action is now

$$S = -\frac{k}{\pi} \sqrt{2\lambda} c_2^{3/2} \int \Phi^6 \left[ \sqrt{-G} \sqrt{1 + \frac{(B+W)^2}{c_2 \Phi^4}} - 1 \right] d^3x. \quad (3.29)$$

Next, let us consider treating  $B$  as an independent field and adding a Lagrange multiplier term to  $S$ :

$$S' = \frac{k}{4\pi} \int F \wedge (B - d\sigma) = \frac{k}{4\pi} \int \varepsilon^{\mu\nu\rho} F_{\mu\nu} (B_\rho - \partial_\rho \sigma) d^3x \quad (3.30)$$

Solving the equation of motion for the Lagrange multiplier  $F$  restores the original action. On the other hand, the  $\sigma$  equation of motion implies that  $dF = 0$  so that we can write  $F = dA$ . The  $B$  equation of motion now becomes

$$\frac{k}{4\pi} \varepsilon^{\mu\nu\rho} F_{\mu\nu} = \frac{k}{\pi} \sqrt{2c_2 \lambda} \sqrt{-G} \frac{(B^\rho + W^\rho) \Phi^2}{\sqrt{1 + \frac{(B+W)^2}{c_2 \Phi^4}}}. \quad (3.31)$$

Squaring both sides one finds that

$$\left( 1 + \frac{F^2}{16\lambda c_2^2 \Phi^8} \right) \left( 1 + \frac{(B+W)^2}{c_2 \Phi^4} \right) = 1. \quad (3.32)$$

This allows one to solve the previous equation for  $B_\mu + W_\mu$ . A nice choice of normalization is  $c_2 = (8\lambda)^{-1}$ , which gives

$$(1 + 4\lambda \Phi^{-8} F^2) (1 + 8\lambda \Phi^{-4} (B+W)^2) = 1. \quad (3.33)$$

We will compare these equations to ones for a D2-brane probe in the next subsection. The transformation we have made equates the Bianchi identity for  $F$  with the equation of motion for  $B$ . Thus, the vanishing of the divergence of the right-hand side of eq. (3.31) is the latter equation.



### 3.3 The D2-brane in $AdS_4 \times CP^3$

Following ABJM [27], it is instructive to consider a type IIA superstring theory background that corresponds to a certain limit of the M-theory one. String theory in this background has a string coupling constant  $g_s$  and a perturbative string expansion. The appropriate background geometry,  $AdS_4 \times CP^3$ , has  $OSp(6|4)$  superconformal symmetry once the fermionic degrees of freedom are included. The bosonic truncation, which is all that will be considered here, only exhibits the  $Sp(4)$  conformal symmetry and the  $SU(4)$  R symmetry.

We need to consider a 2-brane to obtain a three-dimensional conformal field theory, and the only BPS 2-brane in type IIA superstring theory is the D2-brane. This immediately leads to a surprising conclusion. The conformally invariant effective action that will result from treating this D2-brane by the methods of this paper will lead to the inclusion of a dynamical  $U(1)$  gauge field. One usually argues that this is not possible: the kinetic term of such a field is dimension four, but all terms must have dimension three. Certainly, there are no dynamical gauge fields to be found in ABJM theory. The resolution of this paradox, which will be exhibited by formulas shortly, is that the kinetic term of the gauge field will have the form  $F^2/\Phi^2$ , which does have dimension three. This is sensible, because we are considering the theory on the Coulomb branch.

The metric for this problem is

$$ds^2 = R^2 (c_2 \Phi^4 dx \cdot dx + \Phi^{-2} d\Phi^2 + ds_{CP^3}^2), \quad (3.34)$$

where we have copied the  $AdS_4$  expression from eq. (3.1). The radius  $R$  is the same as in eq. (3.19). However, the equality of the M2-brane and D2-brane tensions implies that  $l_p^3 = g_s l_s^3$ . Therefore, expressed in string units, the radius is given by

$$R^6 = 32\pi^2 g_s^2 k N l_s^6. \quad (3.35)$$

Furthermore, since the radius of the M-theory circle is  $R/k$ , which in string units must correspond to  $g_s l_s$ , one deduces that

$$R^2 = 2^{5/2} \pi \sqrt{\lambda} l_s^2 \quad (3.36)$$

and

$$g_s = \sqrt{\pi} (2\lambda)^{5/4} / N. \quad (3.37)$$

As before,  $\lambda = N/k$  is the 't Hooft coupling.

The  $CP^3$  metric can be written in terms of homogeneous coordinates, exhibiting its  $SU(4)$  symmetry, as follows

$$ds_{CP^3}^2 = \frac{dz^A d\bar{z}_A}{|z|^2} - \frac{|\bar{z}_A dz^A|^2}{|z|^4}, \quad (3.38)$$

where  $|z|^2 = z^A \bar{z}_A$ . This formula depends on four complex coordinates, but it describes a six-dimensional manifold. The reason this works is that the formula has a local symmetry under  $z^A \rightarrow \lambda z^A$ , where  $\lambda$  is any nonzero complex function.

In the previous problems we were able to combine the  $\phi^{-2}d\phi^2$  term in the AdS metric with the metric of the compact space in a convenient manner. That is also the case here. The key step is to use the local symmetry to set  $|z| = \phi$ . Doing this, and renaming the coordinate  $\Phi^A$ , gives

$$ds_7^2 = \frac{(d\Phi)^2}{\Phi^2} + ds_{\text{CP}^3}^2 = \frac{d\Phi^2}{\Phi^2} + \frac{d\Phi^A d\bar{\Phi}_A}{\Phi^2} - \frac{|\bar{\Phi}_A d\Phi^A|^2}{\Phi^4} = \frac{d\Phi^A d\bar{\Phi}_A}{\Phi^2} - \frac{[\text{Im}(\bar{\Phi}_A d\Phi^A)]^2}{\Phi^4}. \quad (3.39)$$

This formula still has a local U(1) symmetry given by  $\Phi^A \rightarrow e^{i\alpha} \Phi^A$ . Therefore it describes a seven-dimensional manifold. Since it incorporates the radial AdS coordinate, it is obviously noncompact. Altogether, we have the ten-dimensional metric

$$ds^2 = R^2 (c_2 \Phi^4 dx \cdot dx + ds_7^2) = c_2 R^2 \Phi^4 \left( dx \cdot dx + \frac{ds_7^2}{c_2 \Phi^4} \right). \quad (3.40)$$

The pullback of this metric to the three-dimensional D2-brane world volume is denoted  $c_2 R^2 \Phi^4 G_{\mu\nu}$ . The crucial fact is that the  $G_{\mu\nu}$  obtained here, evaluated in static gauge, is precisely the same expression obtained in eq. (3.27).

We can now write down the D2-brane action. Using the fact that the  $S_2$  cancels the potential term as in the previous examples, it is in static gauge

$$\begin{aligned} S &= -T_{\text{D2}} \int \left( \sqrt{-\det(c_2 R^2 \Phi^4 G_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} - (c_2 R^2 \Phi^4)^{3/2} \right) d^3\sigma \\ &= -\beta \int \Phi^6 \left( \sqrt{-\det(G_{\mu\nu} + \gamma \Phi^{-4} F_{\mu\nu})} - 1 \right) d^3\sigma, \end{aligned} \quad (3.41)$$

where

$$\beta = T_{\text{D2}} R^3 c_2^{3/2} = \frac{\sqrt{2kN}}{\pi} c_2^{3/2}, \quad (3.42)$$

$$\gamma = \frac{2\pi\alpha'}{c_2 R^2} = (2c_2 \sqrt{2\lambda})^{-1}. \quad (3.43)$$

Choosing static gauge, as before, and expanding the action to extract the kinetic terms for the scalar fields  $\Phi^A$  and the U(1) gauge field, one finds

$$-\frac{\beta}{2c_2} \int \text{Re} [\partial^\mu \Phi^A \partial_\mu \bar{\Phi}_A] d^3x - \frac{1}{4} \beta \gamma^2 \int \Phi^{-2} F^{\mu\nu} F_{\mu\nu} d^3x. \quad (3.44)$$

The only freedom in these terms concerns the  $\Phi$  normalization which is encoded in the  $c_2$  dependence. If one chooses

$$c_2 = (8\lambda)^{-1} \quad (3.45)$$

then the kinetic terms become

$$-\frac{k}{4\pi} \int \partial^\mu \text{Re} [\Phi^A \partial_\mu \bar{\Phi}_A] d^3x - \frac{k}{8\pi} \int \Phi^{-2} F^{\mu\nu} F_{\mu\nu} d^3x. \quad (3.46)$$

The determinant in the action can be evaluated by methods similar to those of the previous section. One obtains

$$\sqrt{-\det(G_{\mu\nu} + \gamma \Phi^{-4} F_{\mu\nu})} = \sqrt{-G} \sqrt{1 + \frac{1}{2} \gamma^2 \Phi^{-8} F \cdot F}, \quad (3.47)$$

where  $F \cdot F = G^{\mu\rho} G^{\nu\lambda} F_{\mu\nu} F_{\rho\lambda}$ .

Let us now treat  $F$  as an independent field and add a Lagrange multiplier term to ensure that  $F = dA$  in analogy with the discussion in the previous section. Then the action becomes

$$-\beta \int \Phi^6 \left( \sqrt{-G} \sqrt{1 + \frac{1}{2} \gamma^2 \Phi^{-8} F \cdot F} - 1 \right) d^3x + \frac{k}{4\pi} \int [B \wedge (F - dA) + W \wedge F] \quad (3.48)$$

The  $B$  equation gives  $F = dA$  and the  $A$  equation gives  $dB = 0$ , which allows us to set  $B = d\sigma$ . The  $WF$  term must be part of  $S_2$  that we have omitted until now. It is required in order that the  $F$  variation produces the combination  $B + W$ , which is required for consistency with the previous section. In fact, ABJM point out that the ten-dimensional background contains a RR two-form field strength  $F_2 \sim kJ$ , where  $J = dW$ . So this term is actually a known part of  $S_2$ .

The  $F$  equation of motion is

$$\frac{k}{8\pi\sqrt{-G}} \varepsilon^{\mu\nu\rho} (B_\mu + W_\mu) = \frac{1}{2} \frac{\beta \gamma^2 \Phi^{-2} F^{\nu\rho}}{\sqrt{1 + \frac{1}{2} \gamma^2 \Phi^{-8} F \cdot F}}. \quad (3.49)$$

Squaring this gives

$$(1 + \frac{1}{2} \gamma^2 \Phi^{-8} F \cdot F) (1 + c_2^{-1} \Phi^{-4} (B + W)^2) = 1. \quad (3.50)$$

For the nice normalization choice  $c_2 = (8\lambda)^{-1}$  this becomes

$$(1 + 4\lambda \Phi^{-8} F \cdot F) (1 + 8\lambda \Phi^{-4} (B + W)^2) = 1 \quad (3.51)$$

in perfect agreement with what we found in the previous subsection.

### 3.4 Summary

To summarize, we have found two equivalent formulations of the ABJM probe-brane action that are related by a duality transformation.<sup>5</sup> The D2-brane formulation has the field content of an abelian super Yang-Mills theory, and the M2-brane formulation has the field content of an abelian superconformal Chern-Simons theory. The only information that is lacking in the D2-brane derivation, by itself, is the fact that the string coupling constant should take the form  $g_s = \sqrt{\pi}(2N/k)^{5/4}/N$ , where  $k$  and  $N$  are positive integers.

For the preferred normalization choice  $c_2 = (8\lambda)^{-1}$  the complete D2-brane action is

$$S = -\frac{k}{16\pi\lambda} \int \Phi^6 \left( \sqrt{-\det(G_{\mu\nu} + \sqrt{8\lambda} \Phi^{-4} F_{\mu\nu})} - 1 \right) d^3\sigma + \frac{k}{4\pi} \int W \wedge F, \quad (3.52)$$

where  $F = dA$ , and  $W$  and  $G_{\mu\nu}$  are defined in eqs. (3.25) and (3.27). Since  $k \sim \lambda^{1/4}/g_s$ , we are finding that the action is  $1/g_s$  times an expression that only involves  $\lambda$ , just as we found for the D3-brane problem. As before, this suggests that the loop expansion of the HEA corresponds to the topological expansion of the nonabelian theory. We have conjectured

---

<sup>5</sup>This is analogous to what was demonstrated for flat spacetime backgrounds in [21, 22].

that the case of relevance to the  $U(2)_k \times U(2)_{-k}$  HEA is  $N = 1$ , which implies  $\lambda = 1/k$ . The dual M2-brane action, for the same choice of normalization, is

$$S = -\frac{k}{16\pi\lambda} \int \Phi^6 \left[ \sqrt{-\det \left( \eta_{\mu\nu} + 8\lambda \frac{\text{Re} [D_\mu \Phi^A D_\nu \bar{\Phi}_A]}{\Phi^6} \right)} - 1 \right] d^3x + \frac{k}{4\pi} \int A \wedge dB, \quad (3.53)$$

where  $B$  is the connection used in the definition of the covariant derivatives. By defining  $A^\pm = (A \pm B)/2$ , the last term can be reexpressed as the sum of the two  $U(1)$  Chern-Simons terms with levels  $k$  and  $-k$ .

One interesting problem is the study of soliton solutions. We expect to find a soliton solution that can be interpreted as the fundamental string ending on the D2-brane. This should describe the massive fields of the ABJM theory that have been integrated out. In addition, there should be an instanton that can be interpreted (in the Euclideanized theory) as a D0-brane ending on the D2-brane. It is analogous to the monopole of the four-dimensional theory, which corresponds to a D1-brane ending on a D3-brane.

## 4 The M5-brane in $AdS_7 \times S^4$

In this case the  $AdS_7$  radius is twice the  $S^4$  radius  $R$ . So the 11-dimensional metric is

$$ds^2 = (2R)^2 (v^2 dx^2 + v^{-2} dv^2) + R^2 d\Omega_4^2 = R^2 (c_5 \phi dx^2 + \phi^{-2} d\phi^2 + d\Omega_4^2). \quad (4.1)$$

Here we have made the change of variables  $v = \sqrt{c_5 \phi}/2$ . This is motivated by the fact that  $v$  has dimension 1, whereas a scalar field in six dimensions should have dimension 2. This can be rewritten in the form

$$ds^2 = R^2 (c_5 \phi dx^2 + \phi^{-2} d\phi^I d\phi^I). \quad (4.2)$$

Here  $\phi^I$  is a five-component vector of length  $\phi$ . The radius  $R$  is given in terms of the 11-dimensional Planck length  $l_p$  by

$$R^3 = \pi N l_p^3. \quad (4.3)$$

Therefore, we have (in static gauge)

$$S_1 = -T_{M5} R^6 c_5^3 \int \phi^3 \sqrt{-\det \left( \eta_{\mu\nu} + \frac{\partial_\mu \phi^I \partial_\nu \phi^I}{c_5 \phi^3} + i \tilde{H}_{\mu\nu} \right)} d^6x. \quad (4.4)$$

Since  $T_{M5} = 2\pi/(2\pi l_p)^6$ , the coefficient of  $S_1$  simplifies to

$$T_{M5} R^6 = \frac{N^2}{32\pi^3}. \quad (4.5)$$

The  $\tilde{H}$  term encodes information about the world-volume field that is a two-form with a self-dual three-form field strength. The knowledge of how it does that is not required to understand the discussion that follows. However, here is a very brief description of the

method for doing this that was developed in [20] and [23] and applied to the M5-brane problem in [24] and [25]. The procedure entails treating one spatial dimension differently from the other five and making general coordinate invariance manifest for only five of the six dimensions. The symmetry in the sixth dimension is also there, but it is implemented in a more complicated, less manifest, way. First one defines the five-dimensional restriction of the the six-dimensional three-form  $H = dB$ , which in components is  $H_{\mu\nu\rho}$ . Then one defines the five-dimensional dual  $\tilde{H}^{\mu\nu}$ , and lowers the indices with a six-by-five piece of the six-by-six metric tensor to give  $H_{\hat{\mu}\hat{\nu}}$ , where hatted indices are six dimensional. The matrix appearing in eq. (4.4) is six by six; the hats have been omitted.

Our immediate objective is to check whether  $S_2$  cancels the leading nonderivative term, as in the previous examples. We want  $S_2 = \mu_5 \int A_6$ , but what is the dual potential  $A_6$ ? In general, the equations of motion of M-theory give a rather complicated expression for  $d\star F_4$ , including a term proportional to  $F_4 \wedge F_4$  among others. However, in the special case of the  $AdS_7 \times S^4$  solution, which is known to be an exact solution of M-theory, one has  $d\star F_4 = 0$ . Therefore in this background it is correct to define the dual potential by  $dA_6 = \star F_4$ .

By the same reasoning as in the previous examples, we write  $F_4 = k_5 R^4 \text{vol}(S^4)$ , where  $\text{vol}(S^4)$  is the volume four-form for a four-sphere of unit radius. Then, referring to the metric in eq. (4.1) and inserting a power of  $2\pi l_p$  as before, we see that

$$\star F_4 = \frac{k_5 c_5^3 R^7}{(2\pi l_p)^3} \phi^2 d\phi \wedge dx^0 \wedge \dots \wedge dx^5 = \frac{1}{3} \frac{k_5 c_5^3 R^7}{(2\pi l_p)^3} d(\phi^3 dx^0 \wedge \dots \wedge dx^5). \quad (4.6)$$

and hence

$$S_2 = \frac{1}{3} \frac{k_5 \mu_5 c_5^3 R^7}{(2\pi l_p)^3} \int \phi^3 d^6 x. \quad (4.7)$$

Next we require that

$$\mu_5 \int F_4 = \mu_5 k_5 R^4 \int_{S^4} \text{vol}(S^4) = \mu_5 k_5 R^4 (8\pi^2/3) = 2\pi N. \quad (4.8)$$

This gives

$$\frac{1}{3} \frac{k_5 \mu_5 R^7}{(2\pi l_p)^3} = \frac{N^2}{32\pi^3}. \quad (4.9)$$

We conclude that

$$S = S_1 + S_2 = -\frac{N^2 c_5^3}{32\pi^3} \int \phi^3 \left[ \sqrt{-\det \left( \eta_{\mu\nu} + \frac{\partial_\mu \phi^I \partial_\nu \phi^I}{c_5 \phi^3} + i \tilde{H}_{\mu\nu} \right)} - 1 \right] d^6 x. \quad (4.10)$$

Once again, we find the desired cancellation of the potential.

One of the important motivations for this work was the desire to formulate effective actions for  $(2,0)$  theories in six dimensions on the Coulomb branch. The supersymmetric completion of the bosonic M5-brane action, given above, is a good candidate for the simplest such theory. Its applications are limited, however, because it contains no parameter on which to base a perturbation expansion. However, there should be controlled expansions, at energies small compared to the scale set by the vev of the scalar field. This might justify looking for soliton solutions of the classical action. This would be somewhat analogous

to looking for soliton solutions of eleven-dimensional supergravity. The most interesting soliton candidate is a self-dual string. Such a solution has already been found for the M5-brane probe action in flat 11-dimensional spacetime [28].

In the M5-brane case there is no known analog of ABJM orbifolding, which is the way a perturbative expansion was made possible in the case of the M2-brane theory. Despite its limitations, it is intriguing that there is a candidate for the six-dimensional  $(2,0)$  HEA in the Coulomb phase. After all, there is no known Lagrangian formulation of the unbroken phase. In fact, it is generally assumed that such a formula does not exist. That makes this action all the more significant. The utility of the M5-brane formula may improve after compactification, although this breaks the conformal symmetry explicitly.

## 5 Conclusion

We have conjectured that the world-volume action of a probe  $p$ -brane in a maximally supersymmetric spacetime of the form  $AdS_{p+2} \times S^n$  (or  $3/4$  maximally supersymmetric in the case of  $AdS_4 \times CP^3$ ) can be reinterpreted as the solution to a different problem: finding an explicit formula for the *highly effective action (HEA) of a superconformal field theory in  $p+1$  dimensions on the Coulomb branch*. The bosonic truncations of a few such probe  $p$ -brane world-volume theories were described in detail. The main evidence in support of the conjecture is that the actions incorporate all of the expected symmetries and dualities. In the case of the D3-brane these include  $PSU(2,2|4)$  superconformal symmetry (when fermions are included) and  $SL(2, \mathbb{Z})$  duality. The methodology of the constructions ensures that these properties are built into the formulas. Nevertheless, S-duality was verified explicitly in appendix B.

The actions derived by considering brane probes possess local symmetries: general coordinate invariance and local kappa symmetry. This fact seems quite profound, and it could be an important clue to possible generalizations. We described a specific gauge choice that gives an action with manifest Poincaré and scaling symmetry and the expected field content. It will be accompanied by a fermionic gauge choice for the complete theory with local kappa symmetry. However, it seems reasonable to regard the HEAs with the local symmetries as more fundamental. After all, that is a useful point of view for other gauge theories such as Yang-Mills theory and general relativity.

Important projects for the future are the incorporation of fermions and comparisons of expansions of the resulting formulas with existing results for Coulomb-branch low-energy effective actions in the literature. One feature of HEAs that should be tested is the fact that the probe-brane approach naturally leads to formulas in which only first derivatives of fields appear. Is there some deep reason why there are no higher derivatives? To some extent this is a formalism-dependent question. Some classes of higher-derivative terms can be introduced or removed by field redefinitions. It may be worthwhile to explore the extent to which derivative terms can be removed in this way. Also, derivatives can be moved around by adding total derivative terms to the action. These comparisons will require choosing static gauge, since that is all that standard approaches to the construction of low-energy

effective actions can reproduce. One can also explore whether agreement with existing results require the choice  $N = 1$ .

It should be interesting to construct soliton and instanton solutions of the classical HEAs. In the case of the D3-brane theory, for example, we expect to find a complete  $SL(2, \mathbb{Z})$  multiplet of solitons containing  $W$ , monopole, and dyon supermultiplets. Much more challenging projects will be the construction of multi-soliton and multi-instanton solutions and the exploration of their moduli spaces.

The actions derived in this paper have all of the desired symmetry and duality properties for any positive integer  $N$  (the number of units of flux). We have proposed that the correct choice is  $N = 1$ , even though that is the choice for which the probe approximation is most suspect. We don't understand well the role of the  $N > 1$  actions from the point of view of effective actions. However, if we take the  $N$  dependence at face value, the structure of the formulas suggests that the loop expansion of the D-brane HEAs should correspond to the string loop expansion or equivalently the topological expansion of the Coulomb-branch gauge theory with explicit  $W$  fields. Given our uncertainty about the meaning of  $N$ , this conjecture could be false even if we have correctly identified the  $N = 1$  theory as the HEA. However, if this additional conjecture is correct, it would imply that the classical HEA action and the tree approximation to scattering amplitudes should exhibit dual conformal symmetry in addition to conformal symmetry, and hence have the full Yangian symmetry.

When one incorporates fermions into the brane-probe actions, one will need to decide whether to use component fields or superfields. The formulas will probably be derived first in terms of component fields, but it should be useful for some purposes to recast them in terms of superfields. One could try to do this by modifying the probe-brane analysis or by trying to supersymmetrize the bosonic truncation. There has been some interesting work on superfield formulations of supersymmetric extensions of Born-Infeld theory that retain duality symmetry [29]–[34]. Some of these papers emphasize the relationship between brane-probe actions and Coulomb-branch *low-energy* effective actions.

If our conjecture survives further scrutiny, it will become important to understand the extent to which symmetry and other general considerations determine the HEA. One will also want to understand *why* the world-volume theory of a brane probe should give an HEA. Clearly, the two problems are not completely unrelated. After all, the brane probe provides information about a  $U(N + 1)$  theory that is broken to  $U(N) \times U(1)$  with the brane probe most closely related to the  $U(1)$  factor. However, that interpretation seems to require a large- $N$  approximation. In any case, we have made a precise conjecture for the  $N = 1$  brane-probe action, but not for those with  $N > 1$ .

Another (possibly related) problem concerns the construction of effective actions for higher-rank gauge theories on the Coulomb branch. When the gauge theory has rank  $r > 1$ , the HEA should contain  $r$  massless abelian supermultiplets. It is not clear how to generalize the brane constructions discussed here to address such cases. Presumably, the formulas obtained here with  $N > 1$  are ingredients in these constructions, but it seems likely that additional, more complicated, ingredients are also required.

The analysis in this paper only treated HEAs classically, though this already encodes a lot of quantum information. There is still much to be learned about the classical theories



(inclusion of fermions, structure of tree amplitudes, solitons, possible Yangian symmetry, etc.), so it may be premature to explore their quantization.

In conclusion, we have described a few specific cases in which the world-volume action of a probe brane in an anti de Sitter background with one unit of flux is a good candidate for the highly effective action of a superconformal field theory on the Coulomb branch. Furthermore, in the case of the  $\mathcal{N} = 4$  and ABJM examples, it seems likely that the loop expansion of the HEA corresponds to the topological expansion of the nonabelian gauge theory with explicit  $W$  fields on the Coulomb branch. It will be exciting to see whether these conjectures survive further scrutiny.

*Notes added:*

A. Tseytlin has pointed out that there is evidence of a disagreement between the D3-brane probe action and the  $\mathcal{N} = 4$  SYM Coulomb-branch effective action, which is described in [33]. He claims that the disagreement, which concerns the one-loop  $F^8$  terms, is not removable by a field redefinition. The author is grateful to Tseytlin for bringing this to his attention.

Another proposal for a relationship between gauge theories and probe D-brane actions, which seems to be very different from the one discussed in this paper, has appeared recently [35]. It would be interesting to explore how the two proposals are related.

## Acknowledgments

The author wishes to thank Savdeep Sethi for suggesting that he investigate effective actions for  $(2,0)$  superconformal field theories on the Coulomb branch and for drawing [36, 37] to his attention. He also acknowledges discussions with Hee-Joong Chung, Sergei Gukov, Nicholas Hunter-Jones, Arthur Lipstein, and Wenbin Yan. This work was supported in part by the U.S. Dept. of Energy Grant No. DE-FG03-92-ER40701. The author acknowledges the hospitality of the Aspen Center for Physics, where he began working on this project. The ACP is supported by the National Science Foundation Grant No. PHY-1066293.

## A The AdS Poincaré patch

Consider the description of  $AdS_{p+2}$  with unit radius given by the hypersurface

$$y \cdot y - uv = -1, \quad (\text{A.1})$$

where  $y \cdot y = -(y^0)^2 + \sum_1^p (y^i)^2$  is a Lorentzian product in  $p+1$  dimensions. The Poincaré-patch metric for  $AdS_{p+2}$  with radius  $R$  is then

$$ds^2 = R^2(dy \cdot dy - dudv). \quad (\text{A.2})$$

Defining  $x^\mu = y^\mu/v$  and eliminating  $u = v^{-1} + vx \cdot x$  then gives

$$ds^2 = R^2(v^2 dx \cdot dx + v^{-2} dv^2). \quad (\text{A.3})$$

The coordinate  $x^\mu$  has dimensions of length, and  $v$  has the dimensions of inverse length, as is appropriate for a scalar field in four dimensions.

The hypersurface in eq. (A.1) has  $\text{SO}(p+1, 2)$  symmetry, which corresponds to the conformal symmetry of a  $p$ -brane world-volume theory. Poincaré invariance in  $p+1$  dimensions and the scaling symmetry  $x^\mu \rightarrow \lambda x^\mu$ ,  $v \rightarrow \lambda^{-1}v$  are manifest symmetries of the metric. The only symmetries that are not manifest in the metric of eq. (A.3) are those that correspond to conformal transformations. Infinitesimally, these are given by the hypersurface symmetries  $\delta y^\mu = b^\mu u$ ,  $\delta v = 2b \cdot y$ ,  $\delta u = 0$ . This corresponds to

$$\delta x^\mu = b^\mu(v^{-2} + x \cdot x) - 2b \cdot x x^\mu, \quad \delta v = 2b \cdot x v. \quad (\text{A.4})$$

In order to check the conformal symmetry of the  $AdS$  metric, let us rewrite eq. (A.3) in the form

$$ds^2 = R^2(\eta_{\mu\nu} e^\mu e^\nu + f^2), \quad (\text{A.5})$$

where

$$e^\mu = v dx^\mu \quad \text{and} \quad f = v^{-1} dv. \quad (\text{A.6})$$

In terms of these one-forms the conformal transformations become

$$\delta e^\mu = -2b^\mu v^{-1} f + 2b^\mu x \cdot e - 2x^\mu b \cdot e \quad (\text{A.7})$$

$$\delta f = 2v^{-1} b \cdot e. \quad (\text{A.8})$$

Using these it is easy to verify that  $e \cdot \delta e + f \delta f = 0$ , which proves that the metric is invariant.

## B S-duality of the D3-brane action

As shown in eq. (2.28), the complex parameter  $\tau = \tau_1 + i\tau_2$  is the background value of the complex scalar field of the type IIB supergravity multiplet. It transforms under the  $\text{SL}(2, \mathbb{Z})$  S-duality group in the usual nonlinear fashion,  $\tau \rightarrow (a\tau + b)(c\tau + d)^{-1}$ , where  $a, b, c, d$  are integers satisfying  $ad - bc = 1$ . This symmetry of the D3-brane action is induced from the  $\text{SL}(2, \mathbb{Z})$  symmetry of type IIB superstring theory. We will focus our attention on the  $S$  transformation  $\tau \rightarrow \tau' = -1/\tau$ , or

$$\tau_1 \rightarrow \tau'_1 = -\frac{\tau_1}{|\tau|^2} \quad \text{and} \quad \tau_2 \rightarrow \tau'_2 = \frac{\tau_2}{|\tau|^2}. \quad (\text{B.1})$$

Much of the discussion of duality in the literature is specific to the self-dual point  $\tau = i$ . At this point, the unbroken subgroup of the *classical*  $\text{SL}(2, \mathbb{R})$  duality group is  $\text{SO}(2)$ . S-duality then corresponds to a rotation by  $\pi/2$ , while a rotation by  $\pi$  correspond to sending the fields to their negatives. We will consider the case of arbitrary  $\tau$  (with  $\tau_2 > 0$ ), and focus attention on the  $S$  transformation described above.

Let us begin with free Maxwell theory in a background metric  $G_{\mu\nu}$  written in the form

$$S = -\frac{\tau_2}{8\pi} \int F \cdot F \sqrt{-G} d^4x + \frac{\tau_1}{8\pi} \int F \wedge F = \frac{1}{8\pi} \int (\tau_1 F \cdot \tilde{F} - \tau_2 F \cdot F) \sqrt{-G} d^4x, \quad (\text{B.2})$$

where  $G_{\mu\nu}$  is a Lorentzian-signature metric tensor and  $G$  is its determinant.  $F = dA$  is the usual two-form field strength constructed from a one-form potential. Also,

$$F \cdot F = G^{\mu\rho} G^{\nu\lambda} F_{\mu\nu} F_{\rho\lambda} \quad (\text{B.3})$$

and  $\tilde{F}$  is the Hodge dual:

$$\tilde{F}^{\mu\nu} = \frac{\varepsilon^{\mu\nu\rho\lambda} F_{\rho\lambda}}{2\sqrt{-G}}. \quad (\text{B.4})$$

The  $F \cdot \tilde{F}$  term is metric independent.

The Bianchi identity is  $dF = 0$ , and the classical field equation is  $d\tilde{F} = 0$ . The basic idea of electric-magnetic symmetry, or S-duality, is that the symmetry  $\tau \rightarrow -1/\tau$  can be understood by simultaneously passing to the dual potential, thereby interchanging the roles of the Bianchi identity and the equation of motion. To understand how this works, consider the action with an additional Lagrange multiplier term:

$$S = \frac{1}{8\pi} \int \left( \tau_1 F \cdot \tilde{F} - \tau_2 F \cdot F - 2\tilde{H}^{\mu\nu} (F_{\mu\nu} - 2\partial_\mu A_\nu) \right) \sqrt{-G} d^4x. \quad (\text{B.5})$$

Here  $F_{\mu\nu}$  is treated as an independent field. The equation  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  arises as a consequence of solving the  $H$  equation of motion. In this way one returns to the original action.

The  $A$  equation of motion is solved by introducing a dual potential  $A'_\mu$  with  $H_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$ . Furthermore, the  $F$  equation of motion gives

$$\tilde{H}_{\mu\nu} = \tau_1 \tilde{F}_{\mu\nu} - \tau_2 F_{\mu\nu}. \quad (\text{B.6})$$

The inversion of this formula is

$$-\tilde{F}_{\mu\nu} = \tau'_1 \tilde{H}_{\mu\nu} - \tau'_2 H_{\mu\nu}. \quad (\text{B.7})$$

From this it follows that the action

$$S_{\text{dual}} = \frac{1}{8\pi} \int \left( \tau'_1 H \cdot \tilde{H} - \tau'_2 H \cdot H + 2\tilde{F}^{\mu\nu} (H_{\mu\nu} - 2\partial_\mu A'_\nu) \right) \sqrt{-G} d^4x, \quad (\text{B.8})$$

gives rise to exactly the same set of equations. This proves that  $S$  and  $S_{\text{dual}}$ , without the Lagrange multiplier terms (but with  $F = dA$  and  $H = dA'$ ), are equivalent. This proves S-duality for free Maxwell theory in an arbitrary background geometry. Note that the minus sign on the left-hand side of eq. (B.7) has been accounted for by reversing the sign of the Lagrange multiplier term. This analysis is valid at the quantum level. As evidence of this, note that the topological term, which is proportional to  $\tau_1 \int F \wedge F$ , does not contribute to the classical equations of motion, but it plays an important part in the analysis. The same will be true for the more complicated formulas in the remainder of this appendix.

Let us turn next to the verification of S-duality for the D3-brane action. We should simply substitute eq. (2.29) in place of eq. (B.2) and repeat the same steps. The analysis that follows is based on methods introduced in [38]–[40]. (For earlier work on this subject see [41, 42].) The algebra is a bit overwhelming if one tackles the full problem head on. So let us approach it in a few steps. We begin with the somewhat simpler problem given by the action

$$S = \frac{1}{8\pi} \int \left( \tau_1 F \cdot \tilde{F} - 4\Delta(F, \tau_2) - 2\tilde{H}^{\mu\nu} (F_{\mu\nu} - 2\partial_\mu A_\nu) \right) d^4x, \quad (\text{B.9})$$

where

$$\Delta(F, \tau_2) = \sqrt{-\det(\eta_{\mu\nu} + \sqrt{\tau_2} F_{\mu\nu})} = \sqrt{1 + \frac{1}{2}\tau_2 F \cdot F - \frac{1}{16}\tau_2^2 (F \cdot \tilde{F})^2}. \quad (\text{B.10})$$

As before, the  $H$  equation implies that  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , and the  $A$  equation is solved by writing  $H_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$ .

The  $F_{\mu\nu}$  field equation is

$$\tilde{H}^{\mu\nu} = R^{\mu\nu}(F, \tau) \equiv \tau_1 \tilde{F}^{\mu\nu} + \frac{\tau_2 F^{\mu\nu} - (1/4)\tau_2^2 \tilde{F}^{\mu\nu} F \cdot \tilde{F}}{\Delta(F, \tau_2)}. \quad (\text{B.11})$$

Then the remarkable theorem, which is the essential step in the proof of S-duality, is that this equation can be solved for  $F$  giving

$$\tilde{F}^{\mu\nu} = -R^{\mu\nu}(H, \tau'). \quad (\text{B.12})$$

Before discussing the proof of this formula, let us make clear what it implies. Just as in the preceding discussion of the free theory, it implies that

$$S_{\text{dual}} = \frac{1}{8\pi} \int \left( \tau'_1 H \cdot \tilde{H} - 4\Delta(H, \tau'_2) + 2\tilde{F}^{\mu\nu} (H_{\mu\nu} - 2\partial_\mu A'_\nu) \right) d^4x, \quad (\text{B.13})$$

gives exactly the same set of equations of as the original action. This proves that  $S$  and  $S_{\text{dual}}$ , without the Lagrange multiplier terms (but with  $F = dA$  and  $H = dA'$ ), are equivalent.

In principle, eq. (B.12) can be verified by direct substitution, though this is very difficult. The analysis can be simplified as follows: without loss of generality, the Lorentz covariance of the formulas allows one to choose a Lorentz frame such that  $F$  takes the canonical form

$$F_{\mu\nu} = \begin{pmatrix} 0 & E & 0 & 0 \\ -E & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix}. \quad (\text{B.14})$$

Then the only part of  $H$  that contributes is

$$H_{\mu\nu} = \begin{pmatrix} 0 & E' & 0 & 0 \\ -E' & 0 & 0 & 0 \\ 0 & 0 & 0 & B' \\ 0 & 0 & -B' & 0 \end{pmatrix}. \quad (\text{B.15})$$

The six components of eq. (B.11) then simplify to the two formulas

$$B' = \tau_1 B + \tau_2 E \sqrt{\frac{1 + \tau_2 B^2}{1 - \tau_2 E^2}} \quad (\text{B.16})$$

$$E' = \tau_1 E - \tau_2 B \sqrt{\frac{1 - \tau_2 E^2}{1 + \tau_2 B^2}}. \quad (\text{B.17})$$

Therefore the theorem we wish to verify simplifies to the statement that these two equations for  $E$  and  $B$  are solved by

$$-B = \tau'_1 B' + \tau'_2 E' \sqrt{\frac{1 + \tau'_2 B'^2}{1 - \tau'_2 E'^2}} \quad (\text{B.18})$$

$$-E = \tau'_1 E' - \tau'_2 B' \sqrt{\frac{1 - \tau'_2 E'^2}{1 + \tau'_2 B'^2}}. \quad (\text{B.19})$$

With some effort, this can be verified by direct substitution. The special case  $\tau_1 = \tau'_1 = 0$  is easy to verify.

This is not yet the end. The next step is to introduce an arbitrary Lorentzian signature background metric  $G_{\mu\nu}$ , as we discussed for the free theory. We first note that

$$\sqrt{-\det(G_{\mu\nu} + \sqrt{\tau_2} F_{\mu\nu})} = \sqrt{-G} \Delta, \quad (\text{B.20})$$

where

$$\Delta = \sqrt{-\det(\eta_{mn} + \sqrt{\tau_2} F_{mn})} = \sqrt{1 + \frac{1}{2} \tau_2 F \cdot F - \frac{1}{16} \tau_2^2 (F \cdot \tilde{F})^2}. \quad (\text{B.21})$$

Here, we have implicitly introduced a vierbein  $E_\mu^m$ , such that  $G_{\mu\nu} = E_\mu^m \eta_{mn} E_\nu^n$  and used its inverse to construct  $F_{mn} = E_m^\mu E_n^\nu F_{\mu\nu}$ . This means that the metric  $G$  is used to define index contractions and the Hodge dual in the final expression for  $\Delta$ . The previous argument then goes through without any changes.

The action is now

$$S = \frac{1}{8\pi} \int \left( \tau_1 F \cdot \tilde{F} - 4 \sqrt{-\det(\eta_{mn} + \sqrt{\tau_2} F_{mn})} \right) \sqrt{-G} d^4 x, \quad (\text{B.22})$$

but it is still not in the desired final form. The next step is to utilize the freedom to redefine the metric by a Weyl transformation  $G_{\mu\nu} \rightarrow f G_{\mu\nu}$ , which implies that  $F_{mn} \rightarrow f^{-1} F_{mn}$ . This transforms the action to

$$S = \frac{1}{8\pi} \int \left( \tau_1 F \cdot \tilde{F} - 4 f^2 \sqrt{-\det(\eta_{mn} + f^{-1} \sqrt{\tau_2} F_{mn})} \right) \sqrt{-G} d^4 x \quad (\text{B.23})$$

By choosing  $f = \sqrt{2\pi c_3} \phi^2$ , where  $c_3 = \pi/\lambda$ , eq. (2.29) can be recast in the S-duality invariant form of eq. (B.23) for the choice

$$G_{\mu\nu} = f \left( \eta_{\mu\nu} + \frac{\partial_\mu \phi \cdot \partial_\nu \phi}{c_3 \phi^4} \right) = \sqrt{2\pi c_3} \phi^2 \eta_{\mu\nu} + \frac{\sqrt{2\pi}}{\sqrt{c_3} \phi^2} \partial_\mu \phi \cdot \partial_\nu \phi. \quad (\text{B.24})$$

In this analysis of S-duality we have omitted the important force canceling term proportional to  $\int \phi^4 d^4 x$  contributed by  $S_2$ . It does not depend on the gauge field, and therefore it does not effect the argument.

## C The D3-brane in global $\text{AdS}_5 \times \text{S}^5$

The Poincaré patch description of AdS can be extended to a geodesically complete space. This space has a periodic time coordinate  $\theta$ , but we can pass to the covering space by replacing this circle by an infinite line described by the global time coordinate  $t$ . In this description the  $\text{SO}(2)$  subgroup (after the comma) of  $\text{SO}(p+1, 2)$  is replaced by its non-compact covering group  $\mathbb{R}$ . This is the translation symmetry group for the global time coordinate. Let's quickly review the derivation of the formulas.

We first replace eq. (A.1) for the hypersurface by the equivalent formula

$$\sum_{i=1}^{p+1} y_i^2 - t_1^2 - t_2^2 = -1. \quad (\text{C.1})$$

We then pass to spherical coordinates for the  $y$ 's and the  $t$ 's:  $(y, \Omega_p)$  and  $(\tau, \theta)$ . Replacing  $\theta$  by  $t$ , the metric then takes the form

$$ds^2 = R^2 \left( dy_i dy_i - \frac{y^2 dy^2}{1+y^2} - (1+y^2) dt^2 \right) = R^2 \left( \frac{dy^2}{1+y^2} + y^2 d\Omega_p^2 - (1+y^2) dt^2 \right). \quad (\text{C.2})$$

In these coordinates there is no horizon. The metric is nonsingular at  $y = 0$ , which is the center of global AdS.  $y = \infty$  is the boundary of AdS. For the D3-brane we simply add a five-sphere of radius  $R$ , so

$$ds^2 = R^2 \left( \frac{dy^2}{1+y^2} + y^2 d\Omega_3^2 - (1+y^2) dt^2 + d\Omega_5^2 \right). \quad (\text{C.3})$$

Making the change of coordinates  $y = \frac{1}{2}(\lambda^{-1} - \lambda)$  brings the D3-brane metric to the form

$$ds^2 = R^2 \left( \frac{d\lambda \cdot d\lambda}{\lambda^2} + \frac{1}{4}(\lambda^{-1} - \lambda)^2 d\Omega_3^2 - \frac{1}{4}(\lambda^{-1} + \lambda)^2 dt^2 \right). \quad (\text{C.4})$$

As in the discussion of the Poincaré patch description,  $\lambda^I$  is a six-vector that exhibits the  $\text{SO}(6)$  symmetry of the five-sphere in a convenient way. While these coordinates have some appeal, we will work with  $y$  rather than  $\lambda$  in the remainder of this section.

Let us examine the world-sheet action, starting with  $S_1$  as given in eq. (15). The counterpart of eq. (18) is

$$S_1 = -\frac{N}{2\pi^2} \int \sqrt{-\det \left( G_{\alpha\beta} + \sqrt{\pi/(g_s N)} F_{\alpha\beta} \right)} d^4\sigma, \quad (\text{C.5})$$

where  $G_{\alpha\beta}$  is the pullback of the ten-dimensional metric, with the factor  $R^2$  removed:

$$G_{\alpha\beta} = \frac{\partial_\alpha y \partial_\beta y}{1+y^2} + y^2 g_{ij} \partial_\alpha u^i \partial_\beta u^j - (1+y^2) \partial_\alpha t \partial_\beta t + h_{IJ} \partial_\alpha v^I \partial_\beta v^J \quad (\text{C.6})$$

Here we have introduced coordinates for the spheres:  $d\Omega_3^2 = g_{ij} du^i du^j$  and  $d\Omega_5^2 = h_{IJ} dv^I dv^J$ . We do not need to be more explicit than that.

The static gauge choice in this case requires a world sheet of topology  $S^3 \times \mathbb{R}$ . Then one can identify the  $S^3$  coordinates of the world sheet with those of the  $S^3$  in the metric

and the time coordinate of the world sheet with the global time coordinate  $t$ . In the static gauge the metric becomes  $G_{\mu\nu} = K_{\mu\nu} + M_{\mu\nu}$ , where

$$K_{\mu\nu} = \begin{pmatrix} -(1+y^2) & 0 \\ 0 & y^2 g_{ij} \end{pmatrix} \quad (\text{C.7})$$

and

$$M_{\mu\nu} = \frac{\partial_\mu y \partial_\nu y}{1+y^2} + h_{IJ} \partial_\mu v^I \partial_\nu v^J. \quad (\text{C.8})$$

Then, in matrix notation,  $G = K(1 + K^{-1}M)$ , so that

$$\det G = \det K \det(1 + K^{-1}M) \sim \det K(1 + \text{tr}(K^{-1}M) + \dots) \quad (\text{C.9})$$

and

$$\sqrt{-G} \sim \sqrt{-K} \left( 1 + \frac{1}{2} \text{tr}(K^{-1}M) + \dots \right). \quad (\text{C.10})$$

We have dropped the  $U(1)$  gauge field, which is not our concern here. Substituting

$$\sqrt{-K} = y^3 \sqrt{1+y^2} \sqrt{g} \quad (\text{C.11})$$

we obtain

$$S_1 = -\frac{N}{2\pi^2} \int y^3 \sqrt{1+y^2} \left( 1 + \frac{1}{2} \text{tr}(K^{-1}M) + \dots \right) \sqrt{g} d^3 u dt. \quad (\text{C.12})$$

where

$$\text{tr}(K^{-1}M) = -\frac{\dot{y}^2}{(1+y^2)^2} - \frac{h_{IJ} \dot{v}^I \dot{v}^J}{1+y^2} + \frac{g^{ij} \partial_i y \partial_j y}{y^2(1+y^2)} + \frac{g^{ij} h_{IJ} \partial_i v^I \partial_j v^J}{y^2}. \quad (\text{C.13})$$

The lesson we wish to emphasize is that  $S_1$  contributes a potential

$$V_1(y) = \frac{N}{2\pi^2} y^3 \sqrt{1+y^2}. \quad (\text{C.14})$$

Now let us examine  $S_2$ . As before,  $S_2 = \mu_3 k_3 \int_M \text{vol}(AdS_5) + \frac{\chi}{8\pi} \int F \wedge F$ , and the coefficient  $\mu_3 k_3$  is equal to  $2N/\pi^2$ . Since

$$\text{vol}(AdS_5) = y^3 \wedge \text{vol}(S^3) \wedge dt = \frac{1}{4} d(y^4 \text{vol}(S^3) \wedge dt), \quad (\text{C.15})$$

$$S_2 = - \int V_2(y) \sqrt{g} d^3 u dt, \quad (\text{C.16})$$

where

$$V_2(y) = -\frac{N}{2\pi^2} y^4. \quad (\text{C.17})$$

Altogether,

$$V(y) = V_1(y) + V_2(y) = \frac{N}{2\pi^2} (y^3 \sqrt{1+y^2} - y^4). \quad (\text{C.18})$$

Unlike the Poincaré-patch problem, the two terms do not cancel in this case. In fact,  $V(y)$  is a monotonically increasing function of  $y$ , which implies that the D3-brane is attracted to  $y = 0$ , the center of global AdS. It seems reasonable that the tension of a spherical brane would cause it to collapse to a point. (In the case of an anti-D3-brane, the sign of  $V_2$  is reversed, and the collapse force is even stronger.) The interpretation in terms of  $\mathcal{N} = 4$  nonabelian gauge theory is the following: when the theory is placed on an  $S^3$ , it develops a potential that removes the Coulomb branch. The moduli space consists of a single point.



**Open Access.** This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

## References

- [1] J. Scherk and J.H. Schwarz, *Dual models for nonhadrons*, *Nucl. Phys. B* **81** (1974) 118 [[INSPIRE](#)].
- [2] R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in  $AdS_5 \times S^5$  background*, *Nucl. Phys. B* **533** (1998) 109 [[hep-th/9805028](#)] [[INSPIRE](#)].
- [3] J.M. Maldacena, *The large- $N$  limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)] [[INSPIRE](#)].
- [4] R. Kallosh, J. Kumar and A. Rajaraman, *Special conformal symmetry of world volume actions*, *Phys. Rev. D* **57** (1998) 6452 [[hep-th/9712073](#)] [[INSPIRE](#)].
- [5] P. Claus, R. Kallosh, J. Kumar, P.K. Townsend and A. Van Proeyen, *Conformal theory of  $M2$ ,  $D3$ ,  $M5$  and  $D1$ -branes +  $D5$ -branes*, *JHEP* **06** (1998) 004 [[hep-th/9801206](#)] [[INSPIRE](#)].
- [6] R. Kallosh, J. Rahmfeld and A. Rajaraman, *Near horizon superspace*, *JHEP* **09** (1998) 002 [[hep-th/9805217](#)] [[INSPIRE](#)].
- [7] R. Metsaev and A.A. Tseytlin, *Supersymmetric  $D3$ -brane action in  $AdS_5 \times S^5$* , *Phys. Lett. B* **436** (1998) 281 [[hep-th/9806095](#)] [[INSPIRE](#)].
- [8] P. Claus, *Super  $M$ -brane actions in  $AdS_4 \times S^7$  and  $AdS_7 \times S^4$* , *Phys. Rev. D* **59** (1999) 066003 [[hep-th/9809045](#)] [[INSPIRE](#)].
- [9] A.A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, in *The many faces of the superworld*, M.A. Shifman ed., World Scientific, Singapore (1999), [[hep-th/9908105](#)] [[INSPIRE](#)].
- [10] S. Kuzenko and I.N. McArthur, *Quantum metamorphosis of conformal symmetry in  $N = 4$  super Yang-Mills theory*, *Nucl. Phys. B* **640** (2002) 78 [[hep-th/0203236](#)] [[INSPIRE](#)].
- [11] S. Bellucci, E. Ivanov and S. Krivonos,  *$AdS/CFT$  equivalence transformation*, *Phys. Rev. D* **66** (2002) 086001 [*Erratum ibid.* **D 67** (2003) 049901] [[hep-th/0206126](#)] [[INSPIRE](#)].
- [12] O. Aharony, N. Seiberg and Y. Tachikawa, *Reading between the lines of four-dimensional gauge theories*, *JHEP* **08** (2013) 115 [[arXiv:1305.0318](#)] [[INSPIRE](#)].
- [13] C.G. Callan and J.M. Maldacena, *Brane death and dynamics from the Born-Infeld action*, *Nucl. Phys. B* **513** (1998) 198 [[hep-th/9708147](#)] [[INSPIRE](#)].
- [14] G. Gibbons, *Born-Infeld particles and Dirichlet  $p$ -branes*, *Nucl. Phys. B* **514** (1998) 603 [[hep-th/9709027](#)] [[INSPIRE](#)].
- [15] L. Brink and J.H. Schwarz, *Quantum superspace*, *Phys. Lett. B* **100** (1981) 310 [[INSPIRE](#)].
- [16] W. Siegel, *Hidden local supersymmetry in the supersymmetric particle action*, *Phys. Lett. B* **128** (1983) 397 [[INSPIRE](#)].

- [17] M.B. Green and J.H. Schwarz, *Covariant description of superstrings*, *Phys. Lett. B* **136** (1984) 367 [[INSPIRE](#)].
- [18] N. Seiberg, *Supersymmetry and nonperturbative  $\beta$ -functions*, *Phys. Lett. B* **206** (1988) 75 [[INSPIRE](#)].
- [19] M. Born and L. Infeld, *Foundations of the new field theory*, *Proc. Roy. Soc. Lond. A* **144** (1934) 425 [[INSPIRE](#)].
- [20] M. Perry and J.H. Schwarz, *Interacting chiral gauge fields in six-dimensions and Born-Infeld theory*, *Nucl. Phys. B* **489** (1997) 47 [[hep-th/9611065](#)] [[INSPIRE](#)].
- [21] P. Townsend, *The eleven-dimensional supermembrane revisited*, *Phys. Lett. B* **350** (1995) 184 [[hep-th/9501068](#)] [[INSPIRE](#)].
- [22] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys. B* **443** (1995) 85 [[hep-th/9503124](#)] [[INSPIRE](#)].
- [23] J.H. Schwarz, *Coupling a selfdual tensor to gravity in six-dimensions*, *Phys. Lett. B* **395** (1997) 191 [[hep-th/9701008](#)] [[INSPIRE](#)].
- [24] P. Pasti, D.P. Sorokin and M. Tonin, *Covariant action for a  $D = 11$  five-brane with the chiral field*, *Phys. Lett. B* **398** (1997) 41 [[hep-th/9701037](#)] [[INSPIRE](#)].
- [25] M. Aganagic, J. Park, C. Popescu and J.H. Schwarz, *World volume action of the M-theory five-brane*, *Nucl. Phys. B* **496** (1997) 191 [[hep-th/9701166](#)] [[INSPIRE](#)].
- [26] N. Berkovits and J. Maldacena, *Fermionic T-duality, dual superconformal symmetry and the amplitude/Wilson loop connection*, *JHEP* **09** (2008) 062 [[arXiv:0807.3196](#)] [[INSPIRE](#)].
- [27] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena,  *$N = 6$  superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [[arXiv:0806.1218](#)] [[INSPIRE](#)].
- [28] P.S. Howe, N. Lambert and P.C. West, *The selfdual string soliton*, *Nucl. Phys. B* **515** (1998) 203 [[hep-th/9709014](#)] [[INSPIRE](#)].
- [29] J. Bagger and A. Galperin, *A new goldstone multiplet for partially broken supersymmetry*, *Phys. Rev. D* **55** (1997) 1091 [[hep-th/9608177](#)] [[INSPIRE](#)].
- [30] F. Gonzalez-Rey, B. Kulik, I. Park and M. Roček, *Selfdual effective action of  $N = 4$  super Yang-Mills*, *Nucl. Phys. B* **544** (1999) 218 [[hep-th/9810152](#)] [[INSPIRE](#)].
- [31] I. Buchbinder, S. Kuzenko and A.A. Tseytlin, *On low-energy effective actions in  $N = 2$ ,  $N = 4$  superconformal theories in four-dimensions*, *Phys. Rev. D* **62** (2000) 045001 [[hep-th/9911221](#)] [[INSPIRE](#)].
- [32] S.M. Kuzenko and S. Theisen, *Supersymmetric duality rotations*, *JHEP* **03** (2000) 034 [[hep-th/0001068](#)] [[INSPIRE](#)].
- [33] I. Buchbinder, A.Y. Petrov and A.A. Tseytlin, *Two loop  $N = 4$  super Yang-Mills effective action and interaction between D3-branes*, *Nucl. Phys. B* **621** (2002) 179 [[hep-th/0110173](#)] [[INSPIRE](#)].
- [34] S.M. Kuzenko, *Self-dual effective action of  $N = 4$  SYM revisited*, *JHEP* **03** (2005) 008 [[hep-th/0410128](#)] [[INSPIRE](#)].
- [35] F. Ferrari, *Gauge theories, D-branes and holography*, [arXiv:1310.6788](#) [[INSPIRE](#)].

- [36] S. Sethi and M. Stern, *A comment on the spectrum of  $H$  monopoles*, *Phys. Lett. B* **398** (1997) 47 [[hep-th/9607145](#)] [[INSPIRE](#)].
- [37] T. Maxfield and S. Sethi, *The conformal anomaly of  $M5$ -branes*, *JHEP* **06** (2012) 075 [[arXiv:1204.2002](#)] [[INSPIRE](#)].
- [38] A.A. Tseytlin, *Selfduality of Born-Infeld action and Dirichlet three-brane of type IIB superstring theory*, *Nucl. Phys. B* **469** (1996) 51 [[hep-th/9602064](#)] [[INSPIRE](#)].
- [39] M.B. Green and M. Gutperle, *Comments on three-branes*, *Phys. Lett. B* **377** (1996) 28 [[hep-th/9602077](#)] [[INSPIRE](#)].
- [40] M. Aganagic, J. Park, C. Popescu and J.H. Schwarz, *Dual  $D$ -brane actions*, *Nucl. Phys. B* **496** (1997) 215 [[hep-th/9702133](#)] [[INSPIRE](#)].
- [41] M.K. Gaillard and B. Zumino, *Duality rotations for interacting fields*, *Nucl. Phys. B* **193** (1981) 221 [[INSPIRE](#)].
- [42] B. Zumino, *Duality rotations*, in *Quantum structure of space and time*, Cambridge University Press, Cambridge U.K. (1981).